Vibration analysis of functionally graded plates with porosity composed of a mixture of Aluminum (Al) and Alumina (Al₂O₃) embedded in an elastic medium

Saidi Hayat, Sahla Meriem  
Civil Engineering Department, Faculty of Technology, University of Sidi Bel Abbes, Laboratory of Materials and Hydrology (LMH), Algeria.  
hayatsaidi2019@yahoo.fr, meriensahla@gmail.com

ABSTRACT In this scientific work, a new shear deformation theory for free vibration analysis of simply supported rectangular functionally graded plate embedded in an elastic medium is presented. Due to technical problems during the fabrication, porosities can be created inside FGM plate which may lead to reduction in strength of materials. In this investigation the FGM plate are assumed to have a new distribution of porosities according to the thickness of the plate. The elastic medium is modeled as Winkler-Pasternak two parameter models to express the interaction between the FGM plate and elastic foundation. The four unknown shear deformation theory is employed to deduce the equations of motion. The Hamilton’s principle is used to derive the governing equations of motion. The accuracy of this theory is verified by compared the developed results with those obtained using others plate theory. Some examples are performed to demonstrate the effect of changing gradient material, elastic parameters, porosity index, and length to thickness ratios on the fundamental frequency of functionally graded plate.

KEYWORDS. Shear deformation theory; Vibration; Functionally graded plate; Porosity; Frequency.

INTRODUCTION

Functionally graded materials (FGMs) are a type of heterogeneous composite materials that exhibits a continuous variation of mechanical properties from one point to another. The concept of functionally graded material was first considered in Japan in 1984 during a space plane project. Such kind material is produced by mixing two or more materials by a graded distribution of the volume fractions of the constituents [1], the FGM is thus suitable for diverse applications, such as thermal coatings of barrier for ceramic engines, electrical devices, energy transformation, biomedical engineering, optics, etc [2-11].
However, in FGM fabrication, micro voids or porosities can occur within the materials during the process of sintering. This is because of the large difference in solidification temperatures between material constituents [12]. Wattanasakulpong [13], also gave the discussion on porosities happening inside FGM samples fabricated by a multi-step sequential infiltration technique. Therefore, it is important to take into account the porosity effect when designing FGM structures subjected to dynamic loadings [14]. Currently, many functionally graded (FG) plate structures which have been employed for engineering fields led to the development of various plate models to study the static, buckling and vibration responses of FG structures [15-19]. The classical plate theory (CPT) is based on the supposition that straight lines which are normal to the neutral surface before deformation remain straight and normal to the neutral surface after deformation. Since the transverse shear deformation is neglected [20-23], it cannot be suitable for the investigating of moderately thick or thick plates in which transverse shear deformation effects are more important. For FG thick and moderately thick plates; the first-order shear deformation theory (FSDT) has been employed [24-27]. In such formulation, the displacements are linearly varied within the thickness and need a shear correction coefficient to correct the unrealistic distribution of the transverse shear stresses and shear strains across the thickness. To avoid the use of the shear correction coefficient, higher-order shear deformation plate theories (HSDTs) have been developed [28-40].

The purpose of this work to propose a new higher-order shear deformation theory for free vibration response of FG plates with porosity embedded in elastic medium. In this investigation the FGM plate are assumed to have a new distribution of porosity according to the thickness of the plate. The elastic medium is modeled as Winkler-Pasternak two parameter models to express the interaction between the FGM plate and elastic foundation. The four unknown shear deformation theory is employed to deduce the equations of motion from Hamilton’s principle. The accuracy of this theory is verified by compared the developed results with those obtained using others plate theory. Some examples are performed to demonstrate the effect of changing gradient material, elastic parameters, porosity index, and length to thickness ratios on the fundamental frequency of functionally graded plate.

**MATHEMATICAL FORMULATION**

In the current work, a FG simply supported rectangular plate with length, width and uniform thickness equal to $a$, $b$ and $h$ respectively is considered. The geometry of the plate and coordinate system are illustrated in Fig. 1. The material characteristics of FG plate are considered to vary continuously within the thickness of the plate in according to the power law distribution as follows

$$E(z) = E_w + (E_r - E_w) \left( \frac{1}{2} + \frac{z}{h} \right)^k - (E_r - E_w)(1 - e^{-\frac{z}{2h}}) \quad (1a)$$

$$\rho(z) = \rho_w + (\rho_r - \rho_w) \left( \frac{1}{2} + \frac{z}{h} \right)^k - (\rho_r - \rho_w)(1 - e^{-\frac{z}{2h}}) \quad (1b)$$

![Figure 1: Schematic representation of a rectangular FG plate resting on elastic foundation.](image)
where the subscripts \(m\) and \(c\) denote the metallic and ceramic components, respectively; and \(p\) is the power law exponent. The value of \(k\) equal to zero indicates a fully ceramic plate, whereas infinite \(p\) represents a fully metallic plate. Since the influences of the variation of Poisson’s ratio \(\nu\) on the behavior of FG plates are very small [37], it is supposed to be constant for convenience.

\(\xi\) is the factor of the distribution of the porosity according to the thickness of the plate [41].

**Kinematics and Strains**

In this investigation, further simplifying supposition are made to the conventional higher shear deformation theory (HSDT) so that the number of unknowns is reduced. The displacement field of the conventional HSDT is expressed by Saidi et al [40].

\[
\begin{align*}
\alpha(x, y, z, t) &= u_0(x, y, t) - \frac{\partial w_0}{\partial x} + f(z) \frac{\partial \theta_x(x, y, t)}{\partial x} \\
\beta(x, y, z, t) &= v_0(x, y, t) - \frac{\partial w_0}{\partial y} + f(z) \frac{\partial \theta_y(x, y, t)}{\partial y} \\
\gamma(x, y, z, t) &= w_0(x, y, t)
\end{align*}
\]

where the shape function \(f(z)\) is chosen according to Mahi et al [42]:

\[
f(z) = \frac{b}{2} \tanh \left( \frac{2z}{b} \right) - \frac{4}{3cosh^2(1)} \left( \frac{z}{b} \right)
\]

Clearly, the displacement field in Eq. (2) considers only four unknowns \((u_0, v_0, w_0\) and \(\phi\)). The nonzero strains associated with the displacement field in Eq. (2) are:

\[
\begin{align*}
\varepsilon_x &= \varepsilon_x^0 + \frac{k}{k_y} + f(z) \frac{\partial \theta_x}{\partial x} \\
\varepsilon_y &= \varepsilon_y^0 + \frac{k}{k_x} + f(z) \frac{\partial \theta_y}{\partial y} \\
\gamma_{xy} &= \gamma_{xy}^0 + f(z) \frac{\partial \theta_x}{\partial y} + f(z) \frac{\partial \theta_y}{\partial x}
\end{align*}
\]

where

\[
\begin{align*}
\begin{bmatrix}
\varepsilon_x^0 \\
\varepsilon_y^0 \\
\gamma_{xy}^0
\end{bmatrix} &= \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} + \frac{\partial w_0}{\partial x} + \frac{\partial w_0}{\partial y} \\
\begin{bmatrix}
k_x \\
k_y \\
k_{xy}
\end{bmatrix} &= \begin{bmatrix}
-\frac{\partial^2 w_0}{\partial x^2} \\
-\frac{\partial^2 w_0}{\partial y^2} \\
-2 \frac{\partial^2 w_0}{\partial x \partial y}
\end{bmatrix} \\
\begin{bmatrix}
\phi_x \\
\phi_y \\
\phi_{xy}
\end{bmatrix} &= \begin{bmatrix}
\frac{\partial^2 \phi}{\partial x^2} \\
\frac{\partial^2 \phi}{\partial y^2} \\
-2 \frac{\partial^2 \phi}{\partial x \partial y}
\end{bmatrix} \\
\begin{bmatrix}
\gamma_{xy}^0 \\
\gamma_{xy}^0 \\
\gamma_{xy}^0
\end{bmatrix} &= \begin{bmatrix}
\frac{\partial \phi}{\partial x} \\
\frac{\partial \phi}{\partial y} \\
\frac{\partial \phi}{\partial x}
\end{bmatrix}
\end{align*}
\]

and

\[
g(z) = -\frac{df(z)}{dz}
\]
For elastic and isotropic FGMs, the constitutive relations can be expressed as:

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy} \\
\tau_{yz} \\
\tau_{xz}
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & 0 & 0 & 0 \\
C_{12} & C_{22} & 0 & 0 & 0 \\
0 & 0 & C_{66} & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 \\
0 & 0 & 0 & 0 & C_{55}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy} \\
\gamma_{yz} \\
\gamma_{xz}
\end{bmatrix}
\]

where \((\sigma_x, \sigma_y, \tau_{xy}, \tau_{yz}, \tau_{xz})\) and \((\varepsilon_x, \varepsilon_y, \gamma_{xy}, \gamma_{yz}, \gamma_{xz})\) are the stress and strain components, respectively. Using the material properties defined in Eq. (1), stiffness coefficients, \(C_{ij}\), can be written as

\[
C_{11} = C_{22} = \frac{E(z)}{1 - \nu^2}, \quad C_{12} = \frac{v E(z)}{1 - \nu^2}, \quad C_{44} = C_{55} = C_{66} = -\frac{E(z)}{2(1 + \nu)}
\]

**Equation of Motion**

Hamilton’s principle is herein employed to determine the equations of motion:

\[
0 = \int_0^l (\delta U + \delta V - \delta K) \, dt
\]

where \(\delta U\) is the variation of strain energy; \(\delta V\) is the variation of work done; and \(\delta K\) is the variation of kinetic energy.

The variation of strain energy of the plate is computed by

\[
\delta U = \int_A \left[ \sigma_x \delta \varepsilon_x + \sigma_y \delta \varepsilon_y + \tau_{xy} \delta \gamma_{xy} + \tau_{xz} \delta \gamma_{xz} + \tau_{yz} \delta \gamma_{yz} \right] \, dV
\]

\[
= \int_A \left[ N_x \delta \varepsilon_x^0 + N_y \delta \varepsilon_y^0 + N_{xy} \delta \gamma_{xy}^0 + M_x \delta \kappa_x^0 + M_y \delta \kappa_y^0 + M_{xy} \delta \kappa_{xy}^0 + M_{xz} \delta \kappa_x^0 + M_{yz} \delta \kappa_y^0 + M_{xz} \delta \kappa_{xy}^0 + S_x \delta \gamma_x^0 + S_y \delta \gamma_y^0 + S_{xy} \delta \gamma_{xy}^0 \right] \, dA = 0
\]

where \(A\) is the top surface and the stress resultants \(N, M, S\) are defined by

\[
\left( N_x, M_x, M_y \right) = \int_{-h/2}^{h/2} \left( 1, y, f \right) \sigma_i \, dz, \quad (i = x, y) \quad \text{and} \quad \left( S_x, S_y, S_{xy} \right) = \int_{-h/2}^{h/2} \left( \tau_{xy}, \tau_{yz}, \tau_{xz} \right) \, dz
\]

The variation of the potential energy of elastic foundation can be calculated by

\[
\delta V = \int_A f_e \delta w_i \, dA
\]

where \(f_e\) is the density of reaction force of foundation.

For the Pasternak foundation model [43-53],

\[
f_e = K_w \frac{\partial^2 w}{\partial x^2} + K_{S1} \frac{\partial^2 w}{\partial y^2} + K_{S2} \frac{\partial^2 w}{\partial y^2}
\]
where $K_W$ is the modulus of subgrade reaction (elastic coefficient of the foundation) and $K_{S1}$ and $K_{S2}$ are the shear moduli of the subgrade (shear layer foundation stiffness). If foundation is homogeneous and isotropic, we will get $K_{S1} = K_{S2} = K_S$. If the shear layer foundation stiffness is neglected, Pasternak foundation becomes a Winkler foundation.

The variation of kinetic energy of the plate can be expressed as:

$$
\frac{\partial K}{\partial t} = \int_A \left\{ I_0 \left[ u_0 \delta \dot{u}_0 + \dot{u}_0 \delta \ddot{u}_0 + \ddot{u}_0 \delta \dot{u}_0 \right] - I_1 \left[ \frac{\partial \delta \ddot{u}_0}{\partial x} + \frac{\partial \delta \dot{u}_0}{\partial y} + \frac{\partial \delta u_0}{\partial y} \right] + J_1 \left[ \frac{\partial \delta \ddot{u}_0}{\partial x} + \frac{\partial \delta \dot{u}_0}{\partial y} + \frac{\partial \delta u_0}{\partial y} \right] \right\} dA
$$

Substituting Eqs. (9), (11), and (13) into Eq. (8), integrating by parts, and collecting the coefficients of $\delta u_0$, $\delta v_0$, $\delta w_0$, and $\delta \phi$; the following equations of motion are obtained:

$$
\begin{align*}
\delta u_0 : & \quad \frac{\partial N_{x0}}{\partial x} + \frac{\partial N_{y0}}{\partial y} = I_0 \ddot{u}_0 - I_1 \frac{\partial \ddot{u}_0}{\partial x} - J_1 \frac{\partial \ddot{u}_0}{\partial y} \\
\delta v_0 : & \quad \frac{\partial N_{y0}}{\partial x} + \frac{\partial N_{x0}}{\partial y} = I_0 \ddot{v}_0 - I_1 \frac{\partial \ddot{v}_0}{\partial x} - J_1 \frac{\partial \ddot{v}_0}{\partial y} \\
\delta w_0 : & \quad \frac{\partial^2 \mathbf{M}_0}{\partial x^2} + 2 \frac{\partial^2 \mathbf{M}_0}{\partial x \partial y} + \frac{\partial^2 \mathbf{M}_0}{\partial y^2} = f - I_0 \ddot{w}_0 - I_1 \frac{\partial \ddot{w}_0}{\partial x} - J_1 \frac{\partial \ddot{w}_0}{\partial y} - I_2 \nabla^2 \ddot{w}_0 - J_2 \nabla^2 \ddot{\phi} \\
\delta \phi : & \quad \frac{\partial^2 \mathbf{M}_0}{\partial x^2} + 2 \frac{\partial^2 \mathbf{M}_0}{\partial x \partial y} + \frac{\partial^2 \mathbf{M}_0}{\partial y^2} = J_1 \left( \frac{\partial \ddot{u}_0}{\partial x} + \frac{\partial \ddot{v}_0}{\partial y} \right) - J_2 \nabla^2 \ddot{w}_0 - K_2 \nabla^2 \ddot{\phi}
\end{align*}
$$

where $\nabla^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2$ is the Laplacian operator in two-dimensional Cartesian coordinate system.
Substituting Eq. (4) into Eq. (6) and the subsequent results into Eqs. (10), the stress resultants are obtained in terms of strains as following compact form:

\[
\begin{bmatrix}
N \\
M^b \\
M'
\end{bmatrix} = 
\begin{bmatrix}
A & B & B' \\
B & D & D' \\
B' & D' & H' \\
\end{bmatrix} 
\begin{bmatrix}
\varepsilon \\
k^b \\
k'
\end{bmatrix}, \quad S = A' \gamma ,
\]

in which

\[
N = \{N_x, N_y, N_{xy}\}, \quad M^b = \{M_{xx}^b, M_{yy}^b, M_{xy}^b\}, \quad M' = \{M_x', M_y', M_{xy}'\},
\]

\[
\varepsilon = \{\varepsilon_x^0, \varepsilon_y^0, \gamma_{xy}^0\}, \quad k^b = \{k_{xx}^b, k_{yy}^b, k_{xy}^b\}, \quad k' = \{k_x', k_y', k_{xy}'\},
\]

\[
A = \begin{bmatrix}
A_{11} & A_{12} & 0 \\
A_{12} & A_{22} & 0 \\
0 & 0 & A_{66}
\end{bmatrix}, \quad B = \begin{bmatrix}
B_{11} & B_{12} & 0 \\
B_{12} & B_{22} & 0 \\
0 & 0 & B_{66}
\end{bmatrix}, \quad D = \begin{bmatrix}
D_{11} & D_{12} & 0 \\
D_{12} & D_{22} & 0 \\
0 & 0 & D_{66}
\end{bmatrix},
\]

\[
B' = \begin{bmatrix}
B'_{11} & B'_{12} & 0 \\
B'_{12} & B'_{22} & 0 \\
0 & 0 & B'_{66}
\end{bmatrix}, \quad D' = \begin{bmatrix}
D'_{11} & D'_{12} & 0 \\
D'_{12} & D'_{22} & 0 \\
0 & 0 & D'_{66}
\end{bmatrix}, \quad H' = \begin{bmatrix}
H'_{11} & H'_{12} & 0 \\
H'_{12} & H'_{22} & 0 \\
0 & 0 & H'_{66}
\end{bmatrix},
\]

\[
S = \{S_x', S_y', \gamma_{xy}'\}, \quad \gamma = \{\gamma_x^0, \gamma_y^0, \gamma_{xy}^0\}, \quad A' = \begin{bmatrix}
A'_{44} & 0 \\
0 & A'_{55}
\end{bmatrix},
\]

and stiffness components are given as:

\[
\begin{bmatrix}
A_{11} & B_{11} & D_{11} & B'_{11} & D'_{11} & H'_{11} \\
A_{12} & B_{12} & D_{12} & B'_{12} & D'_{12} & H'_{12} \\
A_{66} & B_{66} & D_{66} & B'_{66} & D'_{66} & H'_{66}
\end{bmatrix} = \int_{-h/2}^{h/2} C_{11} \left(1, \tau, \tau^2, f(\tau), \tau f(\tau), \tau^2 f(\tau)\right)^T \begin{bmatrix}
1 \\
1 - \nu \\
\nu \\
2
\end{bmatrix} d\tau ,
\]

\[
\begin{bmatrix}
A_{11}, B_{11}, D_{11}, B'_{11}, D'_{11}, H'_{11} \\
A_{12}, B_{12}, D_{12}, B'_{12}, D'_{12}, H'_{12} \\
A_{66}, B_{66}, D_{66}, B'_{66}, D'_{66}, H'_{66}
\end{bmatrix} = \left( A_{11}, B_{11}, D_{11}, B'_{11}, D'_{11}, H'_{11}, \right),
\]

\[
A'_{44} = A'_{55} = \int_{-h/2}^{h/2} C_{44} [g(\tau)]^2 d\tau ,
\]

**Analytical solution for simply-supported FG plates**

Based on Navier technique, the following expansions of generalized displacements are considered to automatically respect the simply supported boundary conditions:
\[ \begin{bmatrix} \mu_0 \\ V_0 \\ w_0 \\ \phi \end{bmatrix} = \sum_{n=1}^{N} \sum_{m=1}^{N} \begin{bmatrix} U_{mn} e^{i\alpha} \cos(\alpha x) \sin(\beta y) \\ V_{mn} e^{i\alpha} \sin(\alpha x) \cos(\beta y) \\ W_{mn} e^{i\alpha} \sin(\alpha x) \sin(\beta y) \\ X_{mn} e^{i\alpha} \sin(\alpha x) \sin(\beta y) \end{bmatrix} \]

where \( \alpha = m\pi / a \) and \( \beta = n\pi / b \), \( \omega \) is the frequency of free vibration of the plate, \( \sqrt{\lambda} = -1 \) the imaginary unit.

Substituting Eqs. (19) into Eq. (15) and collecting the displacements and acceleration for any values of \( m \) and \( n \), the following problem is obtained:

\[ \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{12} & S_{22} & S_{23} & S_{24} \\ S_{13} & S_{23} & S_{33} & S_{34} \\ S_{14} & S_{24} & S_{34} & S_{44} \end{bmatrix} \begin{bmatrix} m_{11} & 0 & m_{13} & m_{14} \\ 0 & m_{22} & m_{23} & m_{24} \\ m_{13} & m_{23} & m_{33} & m_{34} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{bmatrix} \begin{bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \\ X_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]

(20)

where

\[ S_{11} = A_{11} \alpha^2 + A_{66} \beta^2, \quad S_{12} = \alpha \beta (A_{12} + A_{66}), \quad S_{13} = -\alpha (B_{11} \alpha^2 + B_{12} \beta^2 + 2B_{66} \beta^2), \quad S_{14} = -\alpha (B_{11} \alpha^2 + B_{12} \beta^2 + 2B_{66} \beta^2) \]

(21)

\[ S_{22} = A_{66} \alpha^2 + A_{12} \beta^2, \quad S_{23} = -\beta (B_{11} \beta^2 + B_{12} \alpha^2 + 2B_{66} \alpha^2), \quad S_{24} = -\beta (B_{11} \beta^2 + B_{12} \alpha^2 + 2B_{66} \alpha^2), \]

\[ S_{33} = D_{11} \alpha^4 + 2(D_{12} + 2D_{66}) \alpha^2 \beta^2 + D_{22} \beta^4 + K_s + K_t (\alpha^2 + \beta^2), \quad S_{34} = D_{11} \alpha^4 + 2(D_{12} + 2D_{66}) \alpha^2 \beta^2 + D_{22} \beta^4 \]

\[ S_{44} = H_{11} \alpha^4 + 2(H_{12} + 2H_{66}) \alpha^2 \beta^2 + H_{22} \beta^4 + A_{33} \alpha^2 + A_{44} \beta^2 \]

(22)

Eq. (20) is a general form for buckling and free vibration analysis of FG plates resting on elastic foundations under in-plane loads. The stability problem can be carried out by neglecting the mass matrix while the free vibration problem is achieved by omitting the in-plane loads.

**NUMERICAL EXAMPLES AND DISCUSSION**

In this section various numerical examples are examined to check the accuracy of the present formulation in predicting the free vibration behaviours of simply supported FG plates resting on elastic foundation. Two types of FG plates of Al/Al₂O₃ are employed in this investigation. The material characteristics of FG plates are presented in Tab. 1. For convenience, the following non-dimensional parameters are employed:

\[ \hat{\omega} = \omega b \sqrt{\rho / E} \]

<table>
<thead>
<tr>
<th>Properties</th>
<th>Aluminium (Al)</th>
<th>Alumina (Al₂O₃)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s modulus (GPa)</td>
<td>70</td>
<td>380</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>Mass density kg/m³</td>
<td>2702</td>
<td>3800</td>
</tr>
</tbody>
</table>

Table 1: Material properties employed in the FG plates.

In order to validate the present formulations that predict the vibration of FGM plates, various illustrative examples are presented. The present dimensionless frequency of the square FGM plate without a porosity are compared with those of
Thai and Choi [54] as shown in Tab. 2. It can be observed that the results are in very good agreement with those predicted by Thai and Choi [54].

<table>
<thead>
<tr>
<th>$a / h$</th>
<th>$a / b$</th>
<th>Theory</th>
<th>0</th>
<th>0.5</th>
<th>Power law index ($k$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>0.5</td>
<td>5</td>
<td>Ref \cite{49}</td>
<td>11.3952</td>
<td>11.2331</td>
<td>11.1780</td>
</tr>
<tr>
<td></td>
<td>Ref \cite{49}</td>
<td></td>
<td>11.7257</td>
<td>11.4992</td>
<td>11.4270</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>Ref \cite{49}</td>
<td>11.8246</td>
<td>11.5780</td>
<td>11.5005</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td></td>
<td>11.8246</td>
<td>11.5781</td>
<td>11.5005</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>Ref \cite{49}</td>
<td>16.1728</td>
<td>15.4895</td>
<td>15.1887</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td></td>
<td>16.1735</td>
<td>15.4898</td>
<td>15.1890</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>Ref \cite{49}</td>
<td>16.4249</td>
<td>15.6851</td>
<td>15.3663</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td></td>
<td>16.4251</td>
<td>15.6852</td>
<td>15.3663</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>Ref \cite{49}</td>
<td>28.6467</td>
<td>26.8009</td>
<td>25.7640</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>Ref \cite{49}</td>
<td>32.3893</td>
<td>29.7133</td>
<td>28.3322</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td></td>
<td>32.3937</td>
<td>29.7163</td>
<td>28.3346</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>Ref \cite{49}</td>
<td>33.8869</td>
<td>30.8606</td>
<td>29.3467</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td></td>
<td>33.8882</td>
<td>30.8614</td>
<td>29.3474</td>
</tr>
</tbody>
</table>

Table 2: Dimensionless fundamental frequency $\omega$ of rectangular plates ($k_w = k_s = 100$), $\omega = \frac{\omega}{(a)^2 \sqrt{P_{in}} / E_{in}}$

In order to analyse the effect of porosity on the natural frequency of FGM plates, numerical results are presented in Tabs. 2-6 and graphically plotted in Figs. 2-4.

<table>
<thead>
<tr>
<th>$a / h$</th>
<th>$k$</th>
<th>$\xi=0$</th>
<th>SSDT</th>
<th>Present</th>
<th>$\xi=0.1$</th>
<th>SSDT</th>
<th>Present</th>
<th>$\xi=0.2$</th>
<th>SSDT</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>0.4150</td>
<td>0.4152</td>
<td>0.4208</td>
<td>0.4210</td>
<td>0.4274</td>
<td>0.4276</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>1</td>
<td>0.3204</td>
<td>0.3206</td>
<td>0.3144</td>
<td>0.3146</td>
<td>0.3062</td>
<td>0.3062</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.2892</td>
<td>0.2895</td>
<td>0.2754</td>
<td>0.2756</td>
<td>0.2550</td>
<td>0.2554</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.2665</td>
<td>0.2674</td>
<td>0.2470</td>
<td>0.2480</td>
<td>0.2158</td>
<td>0.2170</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.2554</td>
<td>0.2563</td>
<td>0.2352</td>
<td>0.2364</td>
<td>0.2026</td>
<td>0.2042</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0.1134</td>
<td>0.1134</td>
<td>0.1149</td>
<td>0.1149</td>
<td>0.1167</td>
<td>0.1167</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>0.0963</td>
<td>0.0963</td>
<td>0.09616</td>
<td>0.09616</td>
<td>0.0952</td>
<td>0.09598</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>0.0868</td>
<td>0.0868</td>
<td>0.08498</td>
<td>0.08504</td>
<td>0.08256</td>
<td>0.08256</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.0788</td>
<td>0.0788</td>
<td>0.07480</td>
<td>0.07480</td>
<td>0.06896</td>
<td>0.06896</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.07398</td>
<td>0.07410</td>
<td>0.06858</td>
<td>0.06870</td>
<td>0.05978</td>
<td>0.05988</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.07144</td>
<td>0.07150</td>
<td>0.06616</td>
<td>0.06628</td>
<td>0.05738</td>
<td>0.05754</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0.02908</td>
<td>0.02908</td>
<td>0.02948</td>
<td>0.02948</td>
<td>0.02992</td>
<td>0.02992</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>0.02464</td>
<td>0.02464</td>
<td>0.02460</td>
<td>0.02460</td>
<td>0.02454</td>
<td>0.02454</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>1</td>
<td>0.02222</td>
<td>0.02222</td>
<td>0.02174</td>
<td>0.02174</td>
<td>0.02110</td>
<td>0.02110</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.02018</td>
<td>0.02018</td>
<td>0.01915</td>
<td>0.01915</td>
<td>0.01762</td>
<td>0.01763</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.01910</td>
<td>0.01910</td>
<td>0.01770</td>
<td>0.01771</td>
<td>0.01541</td>
<td>0.01542</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.01847</td>
<td>0.01848</td>
<td>0.01715</td>
<td>0.01715</td>
<td>0.01491</td>
<td>0.01492</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3: The first non-dimensional frequencies $\tilde{\omega}$ of Al/Al2O3 square plate for various porosity parameters, power law indices and thickness ratios ($a=10h$, $n=m=1$, $K_w=K_s=0$)
### Table 4: The first non-dimensional frequencies $\hat{\omega}$ of Al/Al$_2$O$_3$ square plate for various porosity parameters, power law indices and thickness ratios ($a=10h$, $n=m=1$, $K_w=100$, $K_s=0$).

<table>
<thead>
<tr>
<th>$a/h$</th>
<th>$k$</th>
<th>$\xi=0$ SSDT</th>
<th>Present</th>
<th>$\xi=0.1$ SSDT</th>
<th>Present</th>
<th>$\xi=0.2$ SSDT</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.5</td>
<td>0.3702</td>
<td>0.3704</td>
<td>0.3716</td>
<td>0.3716</td>
<td>0.3730</td>
<td>0.3730</td>
</tr>
<tr>
<td>1</td>
<td>0.3380</td>
<td>0.3382</td>
<td>0.3342</td>
<td>0.3342</td>
<td>0.3286</td>
<td>0.3286</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.3096</td>
<td>0.3098</td>
<td>0.2990</td>
<td>0.2992</td>
<td>0.2832</td>
<td>0.2834</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.2806</td>
<td>0.2808</td>
<td>0.2748</td>
<td>0.2758</td>
<td>0.2508</td>
<td>0.2518</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.1162</td>
<td>0.1162</td>
<td>0.1179</td>
<td>0.1180</td>
<td>0.1199</td>
<td>0.1200</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.09100</td>
<td>0.090976</td>
<td>0.08976</td>
<td>0.08976</td>
<td>0.08796</td>
<td>0.08802</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.08362</td>
<td>0.08368</td>
<td>0.08044</td>
<td>0.08044</td>
<td>0.07578</td>
<td>0.07578</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.07952</td>
<td>0.07958</td>
<td>0.07516</td>
<td>0.07528</td>
<td>0.06814</td>
<td>0.06820</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.07728</td>
<td>0.07734</td>
<td>0.07318</td>
<td>0.07324</td>
<td>0.06634</td>
<td>0.06646</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.02976</td>
<td>0.02976</td>
<td>0.03022</td>
<td>0.03022</td>
<td>0.03074</td>
<td>0.03074</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>0.02554</td>
<td>0.02554</td>
<td>0.02558</td>
<td>0.02558</td>
<td>0.02562</td>
<td>0.02564</td>
<td></td>
</tr>
</tbody>
</table>

### Table 5: The first non-dimensional frequencies $\hat{\omega}$ of Al/Al$_2$O$_3$ square plate for various porosity parameters, power law indices and thickness ratios ($a=10h$, $n=m=1$, $K_w=0$, $K_s=10$).

<table>
<thead>
<tr>
<th>$a/h$</th>
<th>$k$</th>
<th>$\xi=0$ SSDT</th>
<th>Present</th>
<th>$\xi=0.1$ SSDT</th>
<th>Present</th>
<th>$\xi=0.2$ SSDT</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.3844</td>
<td>0.3846</td>
<td>0.3872</td>
<td>0.3872</td>
<td>0.3900</td>
<td>0.3902</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.3544</td>
<td>0.3544</td>
<td>0.3522</td>
<td>0.3522</td>
<td>0.3490</td>
<td>0.3492</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.3282</td>
<td>0.3284</td>
<td>0.3204</td>
<td>0.3206</td>
<td>0.3082</td>
<td>0.3084</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.3110</td>
<td>0.3118</td>
<td>0.2996</td>
<td>0.3004</td>
<td>0.2808</td>
<td>0.2814</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.3032</td>
<td>0.3038</td>
<td>0.2920</td>
<td>0.2928</td>
<td>0.2736</td>
<td>0.2746</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.1188</td>
<td>0.1188</td>
<td>0.1208</td>
<td>0.1208</td>
<td>0.1230</td>
<td>0.1230</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.1033</td>
<td>0.1033</td>
<td>0.1039</td>
<td>0.1039</td>
<td>0.1045</td>
<td>0.1045</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.09492</td>
<td>0.09498</td>
<td>0.09412</td>
<td>0.09418</td>
<td>0.09294</td>
<td>0.09300</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.08148</td>
<td>0.08148</td>
<td>0.08560</td>
<td>0.08566</td>
<td>0.08189</td>
<td>0.08194</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.08448</td>
<td>0.08454</td>
<td>0.08106</td>
<td>0.08112</td>
<td>0.07536</td>
<td>0.07542</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.08256</td>
<td>0.08262</td>
<td>0.07938</td>
<td>0.07952</td>
<td>0.07404</td>
<td>0.07418</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.03042</td>
<td>0.03042</td>
<td>0.03092</td>
<td>0.03092</td>
<td>0.03148</td>
<td>0.03148</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>0.02638</td>
<td>0.02638</td>
<td>0.02652</td>
<td>0.02652</td>
<td>0.02664</td>
<td>0.02664</td>
<td></td>
</tr>
</tbody>
</table>

Table 4: The first non-dimensional frequencies $\hat{\omega}$ of Al/Al$_2$O$_3$ square plate for various porosity parameters, power law indices and thickness ratios ($a=10h$, $n=m=1$, $K_w=100$, $K_s=0$).

Table 5: The first non-dimensional frequencies $\hat{\omega}$ of Al/Al$_2$O$_3$ square plate for various porosity parameters, power law indices and thickness ratios ($a=10h$, $n=m=1$, $K_w=0$, $K_s=10$).
Table 6: The first non-dimensional frequencies $\hat{\omega}$ of Al/Al$_2$O$_3$ square plate for various porosity parameters, power law indices and thickness ratios ($a=10h$, $n=m=1$, $K_w=100$, $K_s=10$).

Tab. 6 present the natural frequencies of FGM plates resting on elastic foundation for different values of porosity parameter ($\xi = 0$, $\xi = 0.1$, $\xi = 0.2$), and elastic foundation parameters. It can be seen that the results are in excellent agreement with those of Sinusoidal plate theory given by Zenkour, it is also concluded that the increase of porosity parameter leads to increase of natural frequency. It can be shown that the frequencies are increasing with the existence of (Winkler and Pasternak parameters).

Figure 2: Variation of the natural frequency of the FGM plates according to the material power index $k$, mode 1, $a=b$.

Figure 3: Influence of thickness ratio on the frequency of the plate FGM, mode 2, $\xi=0$, ($K_w = K_s = 100$).
As the material power index increases for FGM plates, the dimensionless frequency will decrease. The variation curves of the natural frequency of the first mode of various functionally graded plates as a function of material power index parameter “k”, for different values of porosity was presented in Fig. 2. It can be seen that the increase of porosity parameter leads to an increase of the frequency of the first mode.

Fig. 3 shows the influence of thickness ratio, \( \frac{a}{h} \), on the natural frequency of FGM plates \( (\xi = 0) \), the elastic foundation parameters are taken equal to \( (K_w = K_s = 100) \).

It can be seen that the ratio \( \frac{a}{h} \) has a considerable effect on the frequency of the FGM plate, (The later decreases with the increase of this ratio).

![Figure 4: Effect of Pasternak shear modulus parameter on dimensionless frequency of FGM plates, a/h=10, k=2.](image)

Fig. 4 shows the effect of Pasternak parameters on the variation of the dimensionless frequency of FGM plate for different values of porosity. The results show that the frequency increases with the increase of Pasternak parameter and porosity index.

**CONCLUSION**

This work proposes a new higher-order shears deformation theory for free vibration response of FG plates with porosity embedded in elastic medium. In this investigation the FGM plate are assumed to have a new distribution of porosity according to the thickness of the plate. The elastic medium is modeled as Winkler-Pasternak two parameter model to express the interaction between the FGM plate and elastic foundation. The four unknown shear deformation theory is employed to deduce the equations of motion from Hamilton’s principle. The Hamilton’s principle is used to derive the governing equations of motion. The accuracy of this theory is verified by compared the developed results with those obtained using others plate theory. Some examples are performed to demonstrate the effect of changing gradient material, elastic parameters, porosity index, and length to thickness ratios on the fundamental frequency of functionally graded plate.

It has been demonstrated that the present analytical formulation can accurately predict natural frequencies of FG plates with porosity resting on elastic foundation. Also it can be concluded that the effect of volume fraction distributions, slenderness ratio and porosity on the non-dimensional frequency is significant.

**REFERENCES**


